BIFURCATION AND STABILITY OF AN INCOMPRESSIBLE ELASTIC BODY UNDER HOMOGENEOUS DEAD LOADS WITH SYMMETRY

PART II: MOONEY–RIVLIN MATERIALS

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SUMMARY

We study the bifurcation and energy-stability of equilibria of a Mooney–Rivlin body subject to a system of homogeneous dead loads with two of its principal Biot stresses equal in magnitude.

1. Preliminaries

In our previous work (1), we developed a comprehensive set of conditions for a homogeneous, isotropic, incompressible elastic body subject to homogeneous dead loads to be in a homogeneous, energy-minimal state of deformation. In that paper, we emphasized the case in which two of the principal Biot stresses are equal in magnitude.

As in (1), we denote by $\lambda$ and $\mu$ the two independent principal stretches, and define quantities $\eta = \lambda + \mu$, $\xi = \lambda - \mu$, $\xi = \lambda \mu$. The stored-energy function of the material may then be variously represented as

$$\sigma(\mathbf{F}) = \delta(\lambda, \mu) = \delta(\eta, \xi) = \delta(\xi, \xi) = \delta(i_1, i_2),$$

where $i_1, i_2$ are the principal invariants of the left Cauchy–Green tensor.

We denote by $\mathbf{S}$ the homogeneous first Piola–Kirchhoff stress tensor corresponding to the loading device, and by $\Sigma$ the Biot stress tensor. As in (1), we assume that $\mathbf{S}^T \mathbf{S}$ has two equal eigenvalues and that $\mathbf{S}$ may be parametrized by the magnitude $T > 0$ of this eigenvalue, the third eigenvalue being $h(T)^2$ for some non-negative $C^2$ function $h(\cdot)$. We define

$$g(T) = \text{sgn} (\det \mathbf{S}) h(T).$$

If det $\mathbf{S} > 0$ or if det $\mathbf{S} < 0$ and $\mathbf{h}(T) \equiv T$, the Biot stress tensor is given by

$$\Sigma = T[1 - \mathbf{e}(T) \otimes e(T)] + g(T)e(T) \otimes e(T)$$

for some $C^0$ unit vector-valued function $e(\cdot)$.

Otherwise, $\Sigma$ is given by

$$\Sigma = T[e_1 \otimes e_1 - e_2 \otimes e_2] + h(T)e(T) \otimes e(T),$$

where $e_1$, $e_2$ are any pair of mutually orthogonal unit vectors in the plane normal to $e(T)$.

In (1), we identified two types of possible minimizers in the case where $\Sigma$ was given by (1.3). First, there were symmetric minimizers with $\lambda = \mu$. Existence of these equilibria required that

$$\hat{\sigma}_\lambda(\lambda, \lambda) = T - \lambda^{-3}g(T).$$

For such an equilibrium to be a minimizer, it is necessary that

$$[\lambda^3 - 1][T - g(T)] \geq 0,$$

that

$$P(\lambda, T) = \lambda^3 \hat{\sigma}_\lambda(\lambda, \lambda) - \lambda^{-4}g(T) \geq 0,$$

and that

$$Q(\lambda, T) = \lambda^{-4}g(T) - 3\lambda^{-4}g(T) \geq 0.$$

If (1.6) to (1.8) hold with strict inequality, then the symmetric equilibrium is a strict local minimizer of potential energy. In this work we shall be concerned with the case where $g(\cdot)$ is a linear function, thus $g(T) = qT$ for some constant $q \geq -1$.

For such a loading, the foregoing conditions simplify somewhat. Equation (1.5) becomes

$$\{\lambda^3 - q\}^{-1}\lambda^3 \hat{\sigma}_\lambda(\lambda, \lambda) = T(\lambda) = T,$$

while (1.6) becomes

$$(1 - q)\lambda^3 - 1 \geq 0,$$

and (1.8) may be written as

$$\{\lambda^3 - q\}T(\lambda) \geq 0.$$

Thus, for $q < 1$, symmetric solution branches can be minimizers only if $\lambda > 1$, if $T(\cdot)$ is a non-decreasing function of $\lambda$, and if $P(\lambda, T) \geq 0$. For $q > 1$, symmetric solution branches can be minimizers only if $\lambda < 1$, if $T(\cdot)$ is a non-increasing function of $\lambda$, and if $P(\lambda, T) \geq 0$. If $q = 1$, symmetric equilibrium branches with $\lambda > 1$ ($\lambda < 1$) must have $T(\cdot)$ non-decreasing (non-increasing) and $P(\lambda, T) \geq 0$, in order to be minimizers. If $q = 1$, the symmetric equilibrium with $\lambda = 1$ will
be a strict local minimizer if and only if \( T < T_0 = 4[\tilde{\sigma}_1(3,3) + \tilde{\sigma}_2(3,3)] \), say. If \( T > T_0 \), this equilibrium is a non-minimizer.

The second class of equilibria are the asymmetric equilibria. These must satisfy
\[
\tilde{\sigma}_T = T, \quad \tilde{\sigma}_\xi = \xi^{-2}g(T).
\] (1.12)

A solution of (1.12) with \( \eta > 2\xi^3 \) can be a minimizer only if
\[
[\eta - 2\xi^{-1}][T - g(T)] \geq 0,
\] (1.13)
\[
[\xi + \xi^{-2} - \xi^{-1}\eta][T - g(T)]^2 \geq 0,
\] (1.14)
\[
\delta_{\eta\eta} \geq 0,
\] (1.15)
and
\[
\delta_{\eta\xi}[\delta_{\xi\xi} - 2\xi^{-3}g(T)] - \delta_{\xi\xi} \geq 0.
\] (1.16)

We may combine (1.12) to obtain
\[
\tilde{\sigma}_\xi(\eta, \xi) + \xi^{-2}g(\eta, \xi) = 0,
\] (1.17)
which we regard as defining \( \eta \) as a function of \( \xi \). We may then rewrite (1.12), as
\[
\tilde{\sigma}_\xi(\eta(\xi), \xi) = T(\xi) = T.
\] (1.18)

For the linear loading we consider here, (1.16) may be rewritten as
\[
-\{\delta_{\eta\eta} + q\xi^{-2}\delta_{\eta\xi}\} T(\xi) \geq 0.
\] (1.19)

If \( q = 1 \), (1.13) and (1.14) hold with equality. If \( q < 1 \), they require that \( \xi > 1 \), so that minimizing equilibria must lie in the region \( \xi > 1, 2\xi^3 < \eta < \xi^2 + \xi^{-1} \) of the \((\eta, \xi)\)-plane. If \( q > 1 \), they require that \( \xi < 1 \) so that minimizing equilibria must lie in the region
\[
0 < \xi < 1, \quad 2\xi^3 < \eta < \xi^2 + \xi^{-1}
\]
of the \((\eta, \xi)\)-plane.

In (1), we studied the various kinds of bifurcation that a minimizing branch of equilibria could undergo. Local extrema of \( T_\lambda(\cdot) \) corresponded to an encounter between two symmetric solution branches. Zeros of \( P(\lambda, T_\lambda(\lambda)) \) gave a bifurcation from symmetric to asymmetric solutions. Secondary asymmetric-to-asymmetric bifurcations could arise from local extrema of \( T_\lambda(\lambda) \) or from a branching of equation (1.17).

If \( \Sigma \) is given by (1.4), we found in (1) that equilibria must satisfy
\[
\tilde{\sigma}(\xi, \xi) = T, \quad \tilde{\sigma}_\xi(\xi, \xi) + \xi^{-2}h(T) = 0.
\] (1.20)

For such an equilibrium to be a minimizer, we must have \( 0 < \xi < 1 \),
\[
0 < \xi < \xi^{-1} - \xi^2, \quad \delta_{\eta\xi}(\xi, \xi) \geq 0.
\] (1.21)
and
\[ \sigma_{\xi \xi} [\sigma_{\xi \xi} - 2 \xi^{-3} h(T)] - \sigma_{\xi \xi}^2 \geq 0. \] (1.22)

We may combine (1.20)\textsubscript{1,2} to obtain
\[ \sigma_{\xi}(\xi, \xi) + \xi^{-2} h \sigma_{\xi}(\xi, \xi) = 0, \] (1.23)

which we regard as implicitly defining $\xi$ as a function of $\xi$.

We may then rewrite (1.20)\textsubscript{1} as
\[ \sigma_{\xi}(\xi(\xi), \xi) = \bar{T}(\xi) = T. \] (1.24)

For a linear loading of type (1.4), $h(T) = qT$ for some $q > 1$, and (1.22) may be rewritten as
\[ -\{\sigma_{\xi \xi} + q \xi^{-2} \sigma_{\xi \xi}\} \bar{T}'(\xi) \geq 0. \] (1.25)

Bifurcations in this case correspond either to local extrema of $\bar{T}(\cdot)$ or branch points of (1.23).

In this work, we study the case of linear loading and adopt a specific material model, the Mooney–Rivlin model. We choose this model, first because it is widely used in practical applications. Moreover, its mathematical structure is both sufficiently simple to yield explicit results and sufficiently rich to provide examples of the various types of bifurcations discussed in (1). The cases of equi-biaxial loading (that is, $q = 0$) and equi-triaxial loading (that is, $q = 1$) have already been studied in detail for this model ((2), (3), respectively). Here we shall study the case of general $q$, and review the results of (2), (3) in this more general setting.

In (1), we proved that a global energy-minimizer exists for the problem under consideration provided that $\sigma(\cdot)$ satisfies a mild growth condition. The Mooney–Rivlin energy satisfies this growth condition, so existence of a minimizer is guaranteed.

The stored-energy function for a Mooney–Rivlin material is given by
\[ \bar{\sigma}(\xi, \mu) = \frac{1}{2} C_1 \{ \xi^2 + \mu^2 + (\lambda \mu)^{-2} - 3 \} + \frac{1}{2} C_2 \{ (\lambda \mu)^2 + \lambda^{-2} + \mu^{-2} - 3 \}, \]

where $C_1, C_2$ are positive constants. By dividing the potential-energy expression for the problem by $C_2$ (or, equivalently, by choosing units so that $C_2 = 1$), we obtain a simplified problem involving a single material constant $k = C_1/C_2$, and rescaled principal Biot stresses. This is the problem we shall consider henceforth. Thus, our specific forms of the stored-energy function will be
\[ \bar{\sigma}(\xi, \mu) = \frac{1}{2} k \{ \xi^2 + \mu^2 + (\lambda \mu)^{-2} - 3 \} + \frac{1}{2} \{ (\lambda \mu)^2 + \lambda^{-2} + \mu^{-2} - 3 \}, \]
\[ \bar{\sigma}(\eta, \xi) = \frac{1}{2} k \{ \eta^2 + \xi^{-2} - 2 \xi - 3 \} + \frac{1}{2} \{ \xi^{-2} \eta^2 + \xi^2 + 2 \xi^{-1} - 3 \}, \]
\[ \bar{\sigma}(\xi, \xi) = \frac{1}{2} k \{ \xi^2 + \xi^{-2} + 2 \xi - 3 \} + \frac{1}{2} \{ \xi^{-2} \xi^2 + \xi^2 + 2 \xi^{-1} - 3 \}. \] (1.26)
For this material, and for the case in which $\Sigma$ is given by (1.3), the function $T(\lambda)$ is given by
\[ T(\lambda) = \left( \lambda^3 - q \right) \left( 1 + k \lambda^{-2} \right) \left( \xi^6 - 1 \right). \] (1.27)

The expression $P(\lambda, T(\lambda))$ takes the form
\[ P(\lambda, T(\lambda)) = -\lambda^{-4} \left( \lambda^3 - q \right)^{-1} \left( \lambda^9 - k \lambda^7 + 2kq \lambda^4 - 3k^2 \lambda - k \lambda + 2q \right) \]
\[ = -\lambda^{-4} \left( \lambda^3 - q \right)^{-1} A(\lambda, k, q), \text{ say}. \] (1.28)

Moreover, the gradients of $\delta(\cdot)$ are given by
\[ \delta_\sigma = \left[ k + \xi^{-2} \right] \eta, \quad \delta_\xi = \left[ \xi - k \right] \left[ 1 + \xi^{-3} \right] - \xi^{-2} \eta^2, \] (1.29)
so that equations (1.17), (1.18) become, respectively,
\[ \eta^2 - q(k \xi + \xi^{-1}) \eta - (\xi - k)(\xi^3 + 1) = 0, \]
\[ T(\xi) = \left[ k + \xi^{-2} \right] \eta(\xi), \] (1.30)
where $\eta(\xi)$ is a root of the quadratic (1.30). In this case we further obtain
\[ \left\{ \delta_{\xi \xi} + \xi^{-2} q \delta_{\xi \eta} \right\} = \xi^{-3} \left( 2q - q(k \xi + \xi^{-1}) \right). \] (1.31)
(This expression arises in equation (1.19).) In the case where $\Sigma$ is given by (1.4), the gradients of $\delta(\cdot)$ are
\[ \delta_\sigma = \left[ k + \xi^{-2} \right] \xi, \quad \delta_\xi = \left[ \xi + k \right] \left[ 1 - \xi^{-3} \right] - \xi^{-3} \xi^2, \] (1.32)
so that equations (1.23), (1.24) may be written, respectively, as
\[ \xi^2 - q(k \xi + \xi^{-1}) \xi - (\xi + k)(\xi^3 - 1) = 0, \]
\[ T(\xi) = \left[ k + \xi^{-2} \right] \xi(\xi). \] (1.33)
where $\xi(\xi)$ is a root of the quadratic (1.33). In this case, we also find that
\[ \left\{ \delta_{\xi \xi} + \xi^{-2} q \delta_{\xi \eta} \right\} = \xi^{-2} \left( 2q - q(k \xi + \xi^{-1}) \right). \] (1.34)
(This expression arises in equation (1.25).)

In the ensuing sections of this paper, we shall look at the various distinct subcases of the problem. In the case in which $\Sigma$ is given by (1.3), the results will largely hinge on the behaviours of the functions $T(\lambda)$ (defined in (1.27)), and $A(\lambda, k, q)$ (defined in (1.28)). Since these functions are quite complicated, we study them in detail in Appendices A and B, respectively.

In section 2 of the work, we consider the case in which $\Sigma$ is given by (1.3), and in which $|q| < 1$. We identify critical values of $q$ and $k$, and in each distinct subcase we identify the bifurcation points and the energy-minimal states over the range of $T$. In section 3, we carry out the same task for the case in which $\Sigma$ is given by (1.3), with $q > 1$. In section 4, we study the special case of equi-triaxial tension. We recover the results of Ball and Schaeffer (3) for this case and study it as a limiting case marking the transition between the regime of section 2 and that of section 3. In section 5, we study the case in which $\Sigma$
is given by (1.4). In section 6, we discuss some numerical results which are presented graphically in Figs 1 to 3.

2. The case $-1 < q < 1$

In this section we explore the existence and status as energy-minimizers of symmetric and non-symmetric equilibria in the case in which $\Sigma$ is given by (1.3) and $-1 < q < 1$. For a symmetric equilibrium with stretch $\lambda$ to exist, we must have $T = T(\lambda)$, where $T(\lambda)$ is defined in (1.9). For such an equilibrium to be a minimizer we must have

$$\lambda \geq 1,$$

$$T_\lambda(\lambda) \geq 0,$$

$$A(\lambda, k, q) \leq 0.$$  \hspace{1cm} (2.1)

We study the behaviour of the function $T(\lambda)$ in Appendix A. There we find that if $-1 < q < q_1 = 0.3125$, the function $T(\lambda)$ is strictly monotone increasing on $(1, \infty)$. To each value of $q$ in the range $(q_1, 1)$ there corresponds a critical value $k_0(q)$ of $k$. If $k < k_0(q)$, $T(\lambda)$ will be strictly monotone increasing for $\lambda > 1$. Otherwise, $T(\lambda)$ will have two turning points in this range.

In Appendix B, we study the function $A(\cdot)$. It follows from this study that for $-1 < q < 1$, $A(\cdot)$ is negative at $\lambda = 1$ and has exactly one zero, which we denote by $\lambda^*(q)$, in the range $(1, \infty)$. The function $A(\cdot)$ is positive for $\lambda > \lambda^*(q)$.

We note that

$$3P(\lambda, T) - Q(\lambda, T) = 2644(\lambda, 1) - 4844(\lambda, 1)$$

$$= 2\lambda^{-6}[k - 3\lambda^2][\lambda^6 - 1].$$  \hspace{1cm} (2.2)

Suppose that $q > q_1$, $k > k_0(q)$ and that $\lambda > 1$ is a turning point of $T(\lambda)$. It follows from a simple calculation that

$$k - 3\lambda^2 = -6\lambda^5m(\lambda^2)^{-1}(\lambda^8 - 3q\lambda^6 + 3\lambda^4 - q).$$  \hspace{1cm} (2.4)

Since $m(\cdot)$ must be negative at such a turning point (see Appendix A), this expression is easily seen to be positive. Thus, in such a situation, $P(\lambda, T) > 0$ at both turning points. Moreover, a simple calculation shows that

$$T(\lambda^*) = 2[k + \xi^{-2}]\xi = \tau(\xi)$$  \hspace{1cm} (2.5)

only if $\xi$ is a zero of $P(\lambda, T)$. Since $\tau(\cdot)$ is a lower bound for $T(\lambda)$ at non-symmetric equilibria, $\tau(\lambda^*(q)^2) = T(\lambda^*(q)^2)$, and $\tau(1) > T(1)$, it follows that if $q_1 < q < 1$, $k > k_0(q)$, bifurcation from a symmetric to a non-symmetric state occurs at a higher value of $T$ than does bifurcation from symmetric to symmetric.

We thus see that if $-1 < q < q_1$ or if $q > q_1$ and $k < k_0(q)$, the symmetric solution branch yields a unique equilibrium with $\lambda > 1$ for each $T > 0$. For values of $T \leq T^* = T(\lambda^*(q))$, this equilibrium will be a strict local minimizer.
of energy. For \( T > T^* \) it will be a non-minimizer. If \( q_1 < q < 1, k > k_0(q) \), then \( T_q(\cdot) \) has two turning points in \((1, \lambda^+(q))\), and there is a range \((T_{\min}, T_{\max})\) of \( T \) on which three symmetric equilibria with \( \lambda > 1 \) exist. The two branches on which \( T_q(\cdot) \) is positive correspond to strict local minimizers of energy in this case. The third is a non-minimizer. A simple energy-comparison calculation shows that there exists a critical value \( T \in (T_{\min}, T_{\max}) \) such that for \( T_{\min} \leq T < T \) the lower-stretch equilibrium has the lesser energy, while for \( T < T \leq T_{\max} \) the higher-stretch equilibrium does.

Next, we investigate the existence of non-symmetric equilibria for the case when \(-1 < q < 1\). By (1.6), such an equilibrium must satisfy

\[
\eta^2 - q(k\xi + \xi^{-1})\eta - (\xi - k)(\xi^3 + 1) = 0. \tag{2.6}
\]

Formally, this quadratic possesses two roots:

\[
\eta = \frac{1}{2} q(k\xi + \xi^{-1}) \pm \Delta, \tag{2.7}
\]

where

\[
4\Delta^2 = q^2(k\xi + \xi^{-1})^2 + 4(\xi - k)(\xi^3 + 1). \tag{2.8}
\]

For \( 0 < q < 1 \),

\[
4\Delta^2 = q^2\xi^2[k - k_-(\xi, q)][k - k_+(\xi, q)]. \tag{2.9}
\]

where

\[
k_{\pm}(\xi, q) = \frac{1}{2}q^{-1}[-(1 - q^2)\xi^2 \pm (1 - q^2)\xi^{-2}]. \tag{2.10}
\]

For fixed \( k, q \) the right-hand side of (2.9) will be positive only if

\[
k > k_+(\xi, q) \quad \text{or} \quad k < k_-(\xi, q). \]

For a non-symmetric equilibrium we must have, in addition, \( \eta > 2\xi^4 \). Suppose that \( q > 0 \), \( k = k_+(\xi, q) \) so that

\[
\eta^* = \eta = q^{-1}[-\xi \pm (1 - q^2)\xi^4 + \xi^{-4}]. \tag{2.11}
\]

For \( \xi \) near zero or \( \infty \) this expression will exceed \( 2\xi^4 \); it can be equal to \( 2\xi^4 \) only if

\[
\xi^3 - 2q^{-1}[1 \pm (1 - q^2)\xi^4 + 1] = 0. \tag{2.12}
\]

Thus, there exist two points, \( \xi < 1 < \xi_0 \), such that

\[
\frac{1}{2}q[k_+(\xi, q)\xi + \xi^{-1}] = 2\xi^4 \tag{2.13}
\]

at \( \xi = \xi_0 \). Moreover, we see that

\[
\frac{1}{2}q[k_+(\xi, q) + \xi] > 2\xi^4 \tag{2.14}
\]

for all \( \xi > 0 \). The functions \( k_+(\xi, q) \) are easily seen to be convex; let \( \tilde{k}_+ \) be their respective minima. For \( k > \tilde{k}_+ \) there exist two values of \( \xi \) such that \( k_-(\xi, q) = k \); denote these by \( \xi_1(k) < \xi_2(k) \). A simple asymptotic analysis shows that for either \( \xi \sim \infty, \xi \sim 0 \), \( \eta^* > 2\xi^4, \eta^- < 2\xi^4 \). If \( \tilde{k}_- < k < k_1(q) = k_-(\xi_1, q) \),
then
\[ \eta^+(\xi_1(k)) = \eta^-(\xi_1(k)) < 2\xi_1(k)^4. \]

If \( k > k_2(q) \) then
\[ \eta^+(\xi_1(k)) = \eta^-(\xi_1(k)) > 2\xi_1(k)^4. \]

Similarly, if \( k_- < k < k_2(q) = k_- (\xi, q) \) then
\[ \eta^+(\xi_2(k)) = \eta^-(\xi_2(k)) < 2\xi_2(k)^4. \]

If \( k_2(q) < k \) then
\[ \eta^-(\xi_2(k)) = \eta^- (\xi_2(k)) > 2\xi_2(k)^4. \]

It is easy to show that if \( k < k_- \), \( \eta^-(1) < 2 \). It follows that in all of these cases the expressions \( \eta^+ - 2\xi^4 \) have a total of two zeros in the range of \( \xi \) for which \( k_- (\xi, q) > 0 \). Suppose now that \( k > k_+ \). Then there will exist two values of \( \xi \), \( \xi_{2i}(k) < \xi_{2j}(k) \), such that \( k_+(\xi_{2j}(k), q) = k_+ \), \( j = 1, 2 \). We know from the results of Appendix B that \( \eta^+ - 2\xi^4 = 0 \), \( \{ P(\xi^4, T(x_4)) = 0 \} \) at exactly two values of \( \xi \) if \( 0 < q < 1 \). These two zeros are already accounted for, and \( \eta^+ > 2\xi^4 \) at \( \xi = \xi_{2j}(k), j = 1, 2 \). It follows that two branches of non-symmetric equilibria \( \eta^\pm \) exist; \( \xi_{2i}(k) < \xi < \xi_{2j}(k) \).

If \( -1 < q < 0 \), the quadratic \( (2.6) \) has no positive root for \( \xi < k \). For \( \xi > k \), \( \eta^\pm > 0 \). It follows from the results of Appendix B that \( \eta^+ - 2\xi^4 \) has precisely one zero in this case. For values of \( \xi \) greater than this critical value \( \eta^+(\xi) \) will constitute a non-symmetric branch of equilibria.

In order for a non-symmetric equilibrium to be an energy minimizer, we must have
\[
\begin{align*}
\xi > 1, \\
\eta < \xi^2 + \xi^{-1}, \\
\pm \Delta T^+(\xi) & \geq 0,
\end{align*}
\]  

where
\[ T^+(\xi) = [k + \xi^{-2}]\eta^+(\xi). \]  

If we recall \((2.11)\), it is easy to see that if \( k_+(\xi, q) = k \), then
\[ \xi^2 + \xi^{-1} - \eta^+ = \xi^2 + \xi^{-1} - \eta^- = (\xi^2 + \xi^{-1})(1 - q^{-1}[1 \pm (1 - q^2)^{1/2}]). \]

and so the expression \( \xi^2 + \xi^{-1} - \eta^\pm (\xi) \) is non-zero at such a point. Moreover, this expression can have a zero only if
\[ (1 - q)(\xi^2 + 1)(k + \xi^{-2}) = 0 \]

at that zero. Thus, the expression has no zeros, \( 0 < q < 1 \), and so, by \((2.17)\), \( k_+(\xi, q) > k \), always negative for equilibria with \( k > k_+(\xi, q) \). Hence, the latter equilibria are non-minimizers.
A routine calculation yields the identity
\[ \pm 2\Delta T_A^\pm(\xi) = q[k + \xi^{-2}][k - 3\xi^{-2}]\eta^\pm + 4k\xi^3 - 3k^2\xi^2 + 2k - 3\xi^{-2} + 4k\xi^{-3}. \] (2.19)

Suppose that \( k > k_2 \) and let
\[ \tilde{T} = T_A^+(\zeta_2(k)) = T_A^-(\zeta_2(k)). \]

It is easy to show that \( T_A^\pm(\cdot) \rightarrow \pm \infty \) as \( \xi \downarrow \zeta_2(k) \), so that at the bifurcation point between the two non-symmetric solution branches both branches satisfy all three of conditions (2.15).

We have not been able to establish analytically what the zeros of \( T_A^\pm(\cdot) \) are. However, inspection of (2.19) reveals that, if \( q > 1 \),
\[ T_A^-(\xi) > 0 \quad \text{for} \quad \xi > \max \left\{ \frac{3k}{4}, \sqrt{\frac{3}{k}} \right\}. \]

We now enumerate the possible bifurcations.

(i) If \( -1 < q \leq 0 \) the symmetric branch is a strict local minimizer for \( 0 < T < T^* \). For \( T > T^* \) it is a non-minimizer. At \( T^* \) the non-symmetric branch of equilibria \( \eta^+ \) bifurcates from the symmetric branch. For sufficiently large \( \bar{\xi} \), \( T_A^+(\cdot) \) is monotone increasing and \( T_A^-(\cdot) \rightarrow \infty \) as \( \xi \rightarrow \infty \). For \( T > T^* \) the minimizing equilibrium lies on this branch of non-symmetric equilibria. If \( T_A^-(\cdot) \) has turning points, we may have to carry out an energy-comparison analysis to identify the global minimizer. (This situation arises for sufficiently small \( k \)
in the case when \( q = 0 \), and was treated in (2).)

(ii) (a) If \( 0 < q < q_1, k < k_2(q) \) the situation is identical to case (i) except that we now have a sharp estimate for the range in which \( T_A^+(\cdot) \) can be non-monotone.

(b) If \( 0 < q < q_1, k > k_2(q) \) the symmetric branch is a strict local minimizer \( 0 < T < T^* \); at \( T = T^* \), \( \bar{\xi} = \lambda^*(q)^2 \), the non-symmetric branch \( \eta^- \) bifurcates from the symmetric branch. At \( \bar{\xi} = \zeta_1 \) such that \( k_+((\zeta_1, q) = k \) the non-symmetric branch \( \eta^- \) bifurcates from the non-symmetric branch \( \eta^+ \). Typically, we expect \( T_A^+(\cdot) \) to be monotone decreasing on \( (\zeta_1, \lambda^*(q)^2) \), \( T_A^-(\cdot) \) to be monotone increasing on \( (\zeta_1, \infty) \), but there is no analytical guarantee of this. There certainly exists a critical value \( T_1 \in (0, T^*) \) such that the symmetrical branch furnishes the global minimizer on \( (0, T_1) \) and a critical value \( T_2 \geq 1 \{ k\xi_1 + \xi^{-1}_1 \} = T^* \) such that the \( \eta^- \) furnishes the global minimizer on \( (T_2, \infty) \). In the event that \( T_A^+(\cdot) \) are non-monotone, the global minimizer can again be identified by energy-comparison arguments for \( T \in (T_1, T_2) \).

(iii) (a) If \( q_1 < q < 1, k > k_2(q) \) the situation is identical with case (ii)(a).

(b) If \( q_1 < q < 1, k_2(q) < k < k_0(q) \) the situation is identical with case (ii)(b).

† A numerical study indicates that \( k_0(q) > k_2(q) \), \( q_1 < q < 1 \).
(c) If \( q_1 < q < 1 \), \( k > k_0(q) \) there are two symmetric--symmetric bifurcation points, one symmetric--non-symmetric \(-\) bifurcation point and one non-symmetric \(-\) non-symmetric \(+\) bifurcation point. If \( T_+(\cdot) \) are non-monotone, the corresponding additional bifurcations will exist. If, as we expect, they are monotone, the lower-stretch symmetric branch furnishes the minimizer for \( T \in (T, T^*) \). The non-symmetric branch \( \eta^- \) gives the minimizer for \( T \in (T^*, T^b) \), and the non-symmetric branch \( \eta^+ \) does so for \( T \in (T^b, \infty) \). Otherwise, we can once again use energy-comparison to identify the global minimizer.

3. The case \( q > 1 \)

In this section, we consider the case in which \( \Sigma \) is given by (1.3) and \( q > 1 \). For a symmetric equilibrium with stretch \( \lambda \) to exist and be a minimizer in this case we must have

\[
\begin{align*}
T &= T_a(\lambda), \\
\lambda &\leq 1, \\
T_a(\lambda) &\leq 0, \\
A(\lambda, k, q) &\geq 0.
\end{align*}
\]

(3.1)

It follows from the results of Appendix A that there exists a critical value \( \tilde{k}(q) \) of \( k \). If \( k \geq \tilde{k}(q) \) then \( T_a(\cdot) \) is monotone decreasing on \((0, 1)\). If, on the other hand, \( k < \tilde{k}(q) \), then \( T_a(\cdot) \) will have two turning points in \((0, 1)\).

We see from the results of Appendix B that there exists a critical value \( k_0(q) \) of \( k \). If \( k < k_0(q) \), \( A(\cdot) \) is positive for all \( \lambda \in (0, \infty) \). If \( k > k_0(q) \) then there exist two values \( \lambda_1 < \lambda_2 \) of \( \lambda \) in \((0, 1)\) such that \( A(\cdot) \) is negative on \((\lambda_1, \lambda_2)\).

Simple calculations assure us that \( \tilde{k}(q) \) is a monotone decreasing function of \( q \) while \( k_0(\cdot) \) is monotone increasing. Since both approach the limiting value 3 as \( q \downarrow 1 \), we conclude that \( \tilde{k}(q) < k_0(q) \) for \( q > 1 \).

For a non-symmetric equilibrium to exist and be a minimizer in this case, equation (2.6) must have a root with \( \eta > 2^{q-1} \) such that

\[
\begin{align*}
\xi < 1, \\
\eta < \xi^2 + \xi^{-1}, \\
\Delta T_a^\pm(\xi) &\geq 0.
\end{align*}
\]

(3.2)

For \( q > 1 \), \( \lambda > 0 \) for all \( \xi \). As in the case of \( 0 < q < 1 \), \( \xi^2 + \xi^{-1} - \eta^+(\xi) \neq 0 \), the left-hand side of (2.18) cannot be zero. It follows that for all \( \xi > 0 \)

\[
\eta^+(\xi) > \xi^2 + \xi^{-1}, \quad \eta^-(\xi) < \xi^2 + \xi^{-1}.
\]

(3.3)

We are now ready to enumerate the various possible bifurcations.

(i) If \( 0 < k < \tilde{k}(q) \) the function \( T_a(\cdot) \) has two turning points. Thus, there exists a range \((T_{\min}, T_{\max})\) of \( T \) on which the three symmetric equilibria with \( \lambda \in (0, 1) \) exist. For \( T < T_{\min} \), \( T > T_{\max} \) there exists a unique symmetric equilibrium with
\( \lambda \in (0, 1) \). The non-symmetric equilibrium branch \( \eta^+(\cdot) \) exists and satisfies \( \eta^+(\xi) > 2\xi^2 \) for all \( \xi \in (0, 1) \), but cannot be a minimizer by \((3.3)\). The function \( \eta^-(\xi) - 2\xi^2 \) is negative for \( \xi \approx 0 \), and has no zeros, so no non-symmetric equilibrium corresponding to \( \eta^- \) exists.

For \( T \in (\bar{T}_{\text{min}}, \bar{T}_{\text{max}}) \) the two branches on which \( T_j(\cdot) < 0 \) are strict local minimizers of energy. A simple energy-comparison calculation reveals the existence of \( \bar{T} \in (\bar{T}_{\text{min}}, \bar{T}_{\text{max}}) \) such that for \( \bar{T}_{\text{min}} < T < \bar{T} \) the higher-stretch equilibrium has the lesser energy while for \( \bar{T} < T < \bar{T}_{\text{max}} \), the lower-stretch equilibrium does.

In this case, then, the only bifurcations that occur are two symmetric-symmetric bifurcations. The higher-stretch symmetric branch furnishes the global minimizer for \( T \in (0, \bar{T}) \); for \( T \in (\bar{T}, \infty) \) the lower-stretch branch does.

(ii) If \( \bar{k}(q) < k < k_\text{c}(q) \) the function \( T_j(\cdot) \) is strictly monotone decreasing on \((0, 1)\). By the same reasoning as before \( \eta^-(\xi) - 2\xi^2 < 0 \) on \((0, 1)\), and the \( \eta^+(\cdot) \)-branch of equilibria cannot be a minimizer. Thus, no bifurcations occur, and the global minimizer of energy is the symmetric equilibrium with \( \lambda \in (0, 1) \).

(iii) If \( k > k_\text{c}(q) \) the function \( T_j(\cdot) \) is strictly monotone decreasing on \((0, 1)\). Now the function \( \eta^-(\xi) - 2\xi^2 \) has two zeros, \( \lambda_2^j, \lambda_3^j \), and is positive on \((\lambda_2^j, \lambda_3^j)\).

The function \( A(\cdot) \) is negative on \((\lambda_1^j, \lambda_2^j)\). Let \( T_j = T_\text{c}(\lambda_j), j = 1, 2 \). For \( T < \bar{T}_2 \), \( T > \bar{T}_1 \), the symmetric solution is a strict local minimizer. For \( \bar{T}_1 > T > \bar{T}_2 \) it is a non-minimizer. Thus, for \( T \) in the latter range, the global minimizer is furnished by a non-symmetric \( \eta^- \) equilibrium. If \( T_\text{c}(\cdot) \) is strictly monotone decreasing on \((\lambda_3^j, \lambda_2^j)\) there exists only one such for each value of \( T \) in this range. If \( T_\text{c}(\cdot) \) is non-monotone, the global minimizer can be identified in this case by direct energy comparison.

4. The case \( q = 1 \)

The case when \( q = 1 \) was treated by Ball and Schaeffer \((3)\). In this instance, the function \( T_j(\cdot) \) is positive and convex, tending to \( \infty \) as \( \lambda \) tends to \( 0, \infty \). If \( k > 3 \), the minimum \( T_\text{c} \) of \( T_j(\cdot) \) occurs at a value of \( \lambda > 1 \); if \( k < 3 \), it occurs in \((0, 1)\). The function \( P(\lambda, T) \) is given by

\[
P(\lambda, T) = k(1 + \lambda^{-6}) + (3k^{-4} - \lambda^2) - \lambda^{-4} T,
\]

and we have

\[
P(\lambda, T_\text{c}(\lambda)) = -\lambda^{-4}(\lambda^3 - 1)[\lambda^{3/2} - k\lambda + 2] .
\]

If \( q = 1 \), the functions \( k_{\pm} \) coalesce and we have

\[
\Delta = \frac{3}{2}\xi |k - 2\xi - \xi^{-2}|.
\]

It follows that

\[
\eta^+ = \begin{cases} \xi^2 + \xi^{-1}, & k < 2\xi + \xi^{-2}, \\ \xi(k - \xi), & k > 2\xi + \xi^{-2}, \end{cases}
\]
while
\[
\eta_+ = \begin{cases} 
\bar{\xi}(k - \xi), & k < 2\bar{\xi} + \xi^{-2}, \\
(\xi^2 + \xi^{-1}), & k > 2\bar{\xi} + \xi^{-2}. 
\end{cases} 
\] (4.5)

The branches with \(\eta = \xi^2 + \xi^{-1}\) correspond to symmetric solutions in which the out-of-plane principal stretch is equal to one of the in-plane principal stretches. Of course, \(\xi^2 + \xi^{-1} - 2\xi^3 \geq 0\) for all \(\xi\). The expression
\[
\xi(k - \xi) - 2\xi^2 = -\xi^2[\xi^2 - k\xi^3 + 2] 
\] (4.6)
can be positive only if \(k > 3\). In this case there will be two values \(\xi_1 < \xi_2\) of \(\xi\) such that \(k = 2\xi + \xi^{-2}\), and it is easy to see that the expression in (4.6) will be positive at both of these. Moreover, it is easy to see that
\[
\frac{\partial}{\partial \xi} \{(k + \xi^{-2})(k\xi - \xi^2)\} = k(k - 2\xi - \xi^{-2}). 
\] (4.7)

For \(k > 3\), let \(\xi_0, \xi_*\) denote the two positive zeros of the expression in (4.6);
\[
\xi_0 < 1 < \frac{1}{2}k < \xi_* < k. 
\]

For \(0 < T < T_m\), the only equilibrium to exist is the symmetric equilibrium with \(\lambda = 1\).

If \(0 < k < 3\), the lower-stretch branch of symmetric equilibria is compatible with the minimality conditions. For \(T_m < T < T_0 = 2(k + 1)\) the higher-stretch symmetric branch has \(Q(\lambda, T) < 0\); for \(T > T_0\) it has \(P(\lambda, T) < 0\) and thus is a non-minimizer for all \(T > T_m\). For \(T > T_0\), \(P(1, T) < 0\), so the equilibrium with \(\lambda = 1\) is a non-minimizer. A simple energy comparison shows that there exists a critical value \(\bar{T}\) of \(T\) such that for \(0 < T < \bar{T}\) the global minimizer is given by the symmetric state with \(\lambda = 1\). For \(T > \bar{T}\) it is given by the symmetric branch with \(T_\delta(\cdot) < 0\) (and its peers with \(\eta = \xi^2 + \xi^{-1}\)). Thus there are two bifurcation points. At \(T = T_m\) two symmetric branches meet. At \(T = T_0\), the higher-stretch symmetric branch encounters the symmetric equilibrium with \(\lambda = 1\).

If \(k > 3\), the higher-stretch branch of symmetric equilibria is compatible with the minimality conditions for \(T_m < T < T_0\). For \(T_m < T < T_0\) the lower-stretch branch has \(Q < 0\); for \(T > T_0\) it has \(P < 0\), and thus is a non-minimizer for all \(T > T_m\). For \(0 < T < T_0\), the symmetric equilibrium with \(\lambda = 1\) is a strict local minimizer. For \(T > T_0\), \(P(1, T) < 0\), so it is a non-minimizer. For \(T_0 < T < T_3 = T_0(\xi_0)\) the higher-stretch symmetric branch satisfies the minimality conditions. For \(T > T_3\) it has \(P < 0\) and so is a non-minimizer. For \(T_3 < T < T_4 = T_0(\xi_1)\) the symmetric branch \(\eta^-\) is compatible with the minimality conditions. At \(T_4\) it encounters the ‘non-symmetric’ branch \(\eta^+ = \xi^2 + \xi^{-1}\) (which is actually a symmetric branch with \(\lambda = \xi^{-1} < 1\)). This branch satisfies the minimality conditions for \(T > T_4\), \(\xi > \xi_2\). The remaining bifurcations represent the same sequence of minimizing states referred to different principal
axes. Thus, the non-symmetric branch \( \eta^+(\xi), \xi_* < \xi < \xi_1 \) connects the 'non-symmetric' branch \( \eta^+ = \xi^2 + \xi^{-1}, \xi < 1 \) (that is, the higher-stretch symmetric branch referred to different principal axes) to the lower-stretch symmetric branch. The asymmetric branch \( \eta^- = \xi(k - \xi), \xi_1 < \xi < \xi_2 \) connects the 'non-symmetric' branch with \( \xi < 1 \) to the 'non-symmetric' branch with \( \xi > 1 \).

As in the case when \( k < 3 \), there exists \( T \in (T_m, T_0) \) such that the symmetric equilibrium with \( \lambda = 1 \) has lower energy than that with \( \lambda > 1 \) for \( T \in (T_m, T) \) while the reverse is true for \( T \in (T, T_0) \).

Thus, for \( 0 < T < T \), the global minimizer is given by the symmetric equilibrium with \( \lambda = 1 \). For \( T < T < T_3 \) it is furnished by the family of symmetric equilibria with \( \lambda > 1 \). For \( T_3 < T < T_4 \), it is given by the family of genuinely non-symmetric equilibria with \( \eta(k) = \xi(k - \xi) \). And finally, for \( T > T_4 \) the energy is minimized by the family of symmetric equilibria with \( \lambda < 1 \).

Notice how this relates to the situation for values of \( q \) near, but not equal to, 1. For \( q \) slightly less than 1, \( k_\pm(\xi, q) > k_\pm(\xi, q) \), and for \( k \) sufficiently large we have four bifurcation points where \( \eta^- \) asymmetric branches encounter \( \eta^- \) asymmetric branches, and two where \( \eta^- \) branches encounter symmetric branches. For \( k \) sufficiently small, we just have two bifurcation points; at these the \( \eta^- \)-branches encounter symmetric branches. All of these branches are 'close' to corresponding branches for the case when \( q = 1 \). However, most of those equilibrium branches for \( q < 1 \) which tend in the limit as \( q \uparrow 1 \) to energy-minimizing branches for the case when \( q = 1 \) are not energy-minimizers for the case when \( q < 1 \). (This is readily seen by comparing the results of section 2 to those presented above.)

A similar observation applies to the case when \( q \sim 1, q > 1 \).

5. The case \( \det \mathbf{S} < 0, |q| > 1 \)

In this section, we consider the cases we have so far omitted, namely that in which \( \Sigma \) is given by (1.3) with \( q = -1 \), and that in which \( \Sigma \) is given by (1.4) with \( q > 1 \).

It follows from the results of (1) that in the latter case, symmetric equilibria cannot be minimizers. In order for a non-symmetric equilibrium to exist, equations (1.9) must hold. In order for such an equilibrium to be a minimizer, the additional conditions

\[
0 < \xi < \xi^{-1} - \xi^2, \quad [2\xi - q(k\xi + \xi^{-1})]T'(\xi) \geq 0
\]

must hold. The discriminant of the quadratic (1.9) is

\[
q^2(k\xi + \xi^{-1})^2 + 4(\xi + k)(\xi^3 - 1) = 4\Delta^2.
\]

A simple calculation shows that this expression will always be positive. Thus, there exist two branches of equilibria:

\[
\xi_\pm(\xi) = \frac{k}{2}q(k\xi + \xi^{-1}) \pm \Delta.
\]
However, a simple calculation shows that

$$\xi^{-1} - \xi^2 - \xi_\pm(\xi) = 0$$  \hspace{1cm} (5.3)

only if

$$(q - 1)k(\xi^3 - 1) = 0. \hspace{1cm} (5.4)$$

It follows from the asymptotic behaviour of $\xi_\pm(\cdot)$ near $\xi = 0$, and the fact that $q > 1$, that the expression on the left-hand side of (5.3) is positive for $\xi_-(\cdot)$, negative for $\xi_+(\cdot)$ at any $\xi \in (0, 1)$. Thus, the $\xi_-(\cdot)$ branch furnishes the minimizer. All that remains unresolved is the question of whether this branch has bifurcation points corresponding to turning points of $T(\cdot)$. One readily shows that

$$-\Delta T'(\xi) = q(k + \xi^{-2})(k - 3\xi^{-2})\xi_- + 4k\xi^3 + 3k^2 - 2k + 3\xi^{-2} + 4k\xi^{-3}. \hspace{1cm} (5.5)$$

The right-hand side of (5.5) must be at least $0$ for a minimizer. It is easily seen to be positive at $\xi = 1$ and near $\xi = 0$. Thus, for sufficiently small values of $T$ and for sufficiently large values of $T$, the $\xi_-(\cdot)$ branch yields a unique equilibrium, which is the global energy-minimizer. Should turning points exist, the minimizer may be identified in regions of non-uniqueness by direct energy comparison, exactly as before.

If $q = -1$, then symmetric equilibria must satisfy

$$T = T(\lambda) = \lambda^3 + k\lambda - 1 - k\lambda^{-2}. \hspace{1cm} (5.6)$$

For an equilibrium of this kind to be a minimizer, it must satisfy $P \geq 0$, $Q \geq 0$, $\lambda \geq 1$. The condition $Q > 0$ is easily seen to hold for $\lambda > 1$, since $T(\cdot)$ is monotone increasing. The function $P$ is positive in this case, precisely if the function $A(\cdot)$ is negative, and if $q = -1$ we have

$$A(\lambda, k, -1) = (\lambda^3 + 1)^2(\lambda^3 - k\lambda - 2). \hspace{1cm} (5.7)$$

For $q = -1$, the roots of the quadratic (1.6) are $\xi(\xi - k)$ and $-(\xi^2 + \xi^{-1})$, so the only branch of positive roots is

$$\eta^-(\eta) = \xi, \xi - k). \hspace{1cm} (5.8)$$

For such a root to be an equilibrium, we must have

$$\eta^[-1] = \xi(\xi - k), \hspace{1cm} (\xi - k) \geq k. \hspace{1cm} (5.9)$$

and for such an equilibrium to be a minimizer we must have

$$\begin{cases}
\xi \geq 1, \\
T(\xi) \geq 0,
\end{cases} \hspace{1cm} (5.10)$$

where, now

$$T(\xi) = k\xi^2 - k\xi^2 + 1 - k\xi^{-1}. \hspace{1cm} (5.11)$$

This is strictly monotone increasing for $\xi \geq k$.

† Up to changes of principal axes.
Thus, the bifurcation picture for this case is extremely simple. If \( \lambda_c \) denotes the positive root of \( \lambda^3 - k \lambda - 2 \), \( \xi_c = \lambda_c^2 \) and \( T_c = T(\lambda_c) \) then we have the following.

(i) For \( 0 < T < T_c \) the global minimizer is the unique symmetric equilibrium. For \( T > T_c \), the symmetric equilibrium is not a minimizer.

(ii) For \( T > T_c \) the energy is minimized by the family of non-symmetric equilibria.

6. Numerical results

The foregoing analysis resolves all issues except that of the monotonicity of \( T(\xi) \). We have studied this question numerically for numerous values of \( q \) and \( k \). Here, we present some representative examples.

Figure 1 follows the minimizing equilibria through the primary and secondary bifurcations in the case where \( k = 10, q = 0.6 \). The solid curve represents the function \( T_{\lambda}(\xi) \), the dotted curve \( T_{\lambda}(\xi) \), and the dashed curve \( T_{\lambda}(\xi) \). For these values of \( k \) and \( q \), \( T_{\lambda}(\xi) \) is known to be monotone increasing for \( \xi > 10 \). It is evident from the graph that \( T_{\lambda}(\xi) \) are strictly monotone for the range of \( \xi \) shown. Thus, the bifurcation behaviour is as expected.

For \( 0 < T < 62.1706 \), the energy minimizer is given by the unique symmetric equilibrium with \( \lambda > 1 \). At \( T = 62.1706 \), \( \xi = 0.6422 \) the primary bifurcation occurs. For \( 62.1706 < T < 270.6091 \) the minimizer is furnished by the family
\( \eta^-(\cdot) \) of non-symmetric equilibria. Since \( T^+_A(\cdot) \) is a monotone decreasing function, no turning-point bifurcations occur on this branch. At \( T = 270.6091 \), \( \xi = 8.9988 \) the secondary bifurcation occurs. For \( T > 270.6091 \), the minimizer is given by the family \( \eta^-(\cdot) \) of non-symmetric equilibria. Since \( T^+_A(\cdot) \) is monotone increasing for \( \xi > 9.9988 \), no turning-point bifurcations occur on this branch either. Thus, the minimizer is determined explicitly for all values of \( T \).

Figure 2 treats the case \( q = 0.6, k = 2 \), and graphs the function \( T^+_A(\cdot) \) for 2.0351 < \( \xi \) < 10. This function is clearly monotone increasing on this range. Thus, no turning-point bifurcations occur on the non-symmetric \( \eta^+(\cdot) \) branch. For \( 0 < T < 6.3952 \) the energy is minimized by the symmetric equilibrium with \( \lambda > 1 \). For \( T > 6.3952 \), the minimizer is given by the family \( \eta^-(\cdot) \) of non-symmetric equilibria. The only bifurcation point involving minimizers is the symmetric–non-symmetric bifurcation at \( T = 6.3952, \xi = 2.0352 \).

Figure 3 illustrates the behaviour of \( T^+_A(\cdot) \) for the case where \( q = 2, k = 15 \). It is clearly a monotone decreasing function for \( 0.084 < \xi < 0.2694 \). Thus, no turning-point bifurcations occur on this branch. The bifurcation picture is therefore complete for this case. For \( 0 < T < 30.358 \), the minimizer is furnished by the symmetric equilibrium with \( \lambda < 1 \). At \( T = 30.358, \xi = 0.2694 \) the primary bifurcation occurs. For \( 30.358 < T < 90.815 \), the minimizer is given by the \( \eta^-(\cdot) \) branch of non-symmetric equilibria. At \( T = 90.815, \xi = 0.084 \) the secondary bifurcation occurs. For \( T > 90.815 \), the symmetric equilibrium with \( \lambda < 1 \) minimizes the energy.
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REFERENCES


APPENDIX A

The behaviour of $T_1(\lambda)$

In this Appendix we consider the behaviour of $T_1(\cdot)$ for fixed values of $k$ and $q$. We begin by reviewing the special case when $q = 1$. It follows from (1.3) that

$$T_1(\lambda) = \lambda^3 + 1 + k\lambda + k\lambda^{-2}$$

(A.1)

in this case. This is easily seen to be a positive, convex function which tends to $\infty$ as $\lambda \to 0, \infty$.

We turn now to $q \neq 1$. By differentiating (1.3) we find that

$$T_1'(\lambda) = (\lambda^3 - q)^{-1}(k\lambda^{-2}[\lambda^2 - 4q\lambda^6 + 5\lambda^5 - 2q] + 3\lambda^2[\lambda^6 - 2q\lambda^3 + 1]).$$

(A.2)

The first bracketed term is a cubic in $\lambda^3$. Thus, we are led to consider the cubic

$$m(z) = z^3 - 4qz^2 + 5z - 2q.$$  

(A.3)

An elementary study of $m(\cdot)$ reveals the following facts.

(i) For $-1 \leq q \leq 0$, $m(\cdot)$ has no positive roots, and is positive on $(0, \infty)$.
(ii) If $0 < q < q_1 = 0.3125\sqrt{10}$, $m(\cdot)$ has exactly one positive root, $z_0 \in (0,1)$; $m(\cdot)$ is negative on $(0,z_0)$ and positive on $(z_0, \infty)$. 
(iii) If $q_1 < q < 1$, $m(\cdot)$ has three positive roots $0 < z_1 < 1 < z_2 < \sqrt{2} < z_3 < 2$; $m(\cdot)$ is negative on $(0,z_1)$ and $(z_2,z_3)$, positive on $(z_1,z_2)$ and $(z_3,\infty)$. 
(iv) If $q > 1$, $m(\cdot)$ has exactly one positive root $z_4 \in (1,\infty)$; $m(\cdot)$ is negative on $(0,z_4)$, positive on $(z_4,\infty)$.

The second bracketed term in (A.2) is a quadratic in $\lambda^2$; we introduce the quadratic

$$r(z) = z^2 - 2qz + 1. \tag{A.4}$$

If $-1 < q < 1$, $r(\cdot)$ is positive on $(0, \infty)$; if $q > 1$, $r(\cdot)$ has two positive roots

$$z_{\pm} = q \pm (q^2 - 1)^{\frac{1}{2}}. \tag{A.5}$$

$r(\cdot)$ is negative on $(z_-,z_+)$, positive on $(0, \infty) \setminus [z_-,z_+]$.

In this paper, we are concerned with states of deformation which are local or global minimizers of potential energy. Accordingly, it follows from the results of (1) that for $-1 < q < 1$ we need only consider the behaviour of $T_1(\lambda)$ for $\lambda > 1$, and that for $q > 1$ we may confine our attention to the range $0 < \lambda \leq 1$. We consider first the case when $q > 1$. The function $T_2(\lambda)$ is positive on $(0,1)$, zero at 1, and tends to $\infty$ as $\lambda \to 0$. We wish to resolve the question whether $T_2(\lambda)$ is monotone decreasing on $(0,1)$ or has turning points. If $q > 1$, $m(\cdot)$ will be negative on $(0,1]$, $r(\cdot)$ will be positive on $(0,z_-)$ and negative on $(z_-,1]$. Recall from (A.2) that the sign of $T_1(\lambda)$ will be the same as that of

$$L(z) = km(z) + 3z^2r(z), \tag{A.6}$$

with $z = \lambda^2$. Clearly, $L(\cdot)$ will be negative on $[z_-,1]$. Moreover, as $z \to 0$, the expression $L(\cdot)$ will be dominated by the term $km(z)$, and so will be negative. However, if $k$ is sufficiently small, it is clear that there will be at least some values of $z$ in the range $(0,z_-)$ such that $L(\cdot)$ is positive. To obtain precise information on this point, we study the function

$$S(z) = \frac{z^2m(z)}{3z^2r(z)}. \tag{A.7}$$

on $(0,z_-)$. This expression is negative on its range, and tends to $-\infty$ as $z \to 0$, $z_-$. At values of $z$ for which $S(z) > -k^{-1}$, $L(z)$ will be positive.

By differentiating (A.7), we find, after some rearrangement, that

$$\frac{6z^2r(z)^2zS'(z)}{3z^2r(z)} = \frac{1}{z^2} \left( 2z^2 + 2 \right) \left[ q - z \right] \left[ q - d(z) \right], \tag{A.8}$$

where the expression $d(z)$ is given by

$$d(z) = \left( 8z^3 + 16z \right)^{-1} \left( z^4 + 18z^2 + 5 \right). \tag{A.9}$$

The function $d(\cdot)$ is easily seen to be monotone decreasing on $(0,\sqrt{10})$, and thus, in particular, on $(0,z_-)$. Furthermore, a simple calculation shows that

$$q - d(z_-) = 3\left( 8z_-^3 + 16z_- \right)^{-1} (z_- - 1)^2. \tag{A.10}$$

Thus, it follows from (A.8) that the expression $S(\cdot)$ has a single critical point in $(0,z_-)$, a maximum, which occurs at the point $z_a$ such that

$$d(z_a) = q.$$

The behaviour of $T_1(\lambda)$ when $q > 1$ will, therefore, be determined by the sign of the quantity $S(z_a) + k^{-1}$. If this quantity is non-positive $T_1(\lambda)$ is monotone decreasing.
on \((0, z^1)\). If, on the other hand, this quantity is positive, there will exist two values of 
z, \(z_1 < z < z_2\) in \((0, z_+)\) such that 
\[ S(z_+) + k^{-1} = 0, \quad x = 1, 2. \]
The function \(T(z)\) will be monotone decreasing on \((0, z^1)\) and \((z^1, 1]\), monotone increasing on \((z^1, z_2^1)\) in this case. We define \(k(q) = -S(z_+)^{-1}\).

We now turn to the case when \(-1 \leq q < 1\). We first observe that if \(-1 \leq q \leq q_1\), 
m(\cdot) is positive on \((1, \infty)\). It follows that \(T(z)\) is monotone increasing on \((1, \infty)\). However, for \(q \in (q_1, 1)\) there will exist a range of values \((z_1, z_2)\) in \((1, \infty)\) of \(z\) for which \(m(\cdot)\) is negative. It is apparent, then, that sufficiently large values of \(k\) will give rise to 
negative values of \(L(\cdot)\) within this range. We consider the function 
\[ D(z) = -S(z)^{-1} \tag{A.11} \]
on the interval \((z_1, z_2)\). The function \(D(\cdot)\) is positive on this range and tends to 
\(\infty\) as \(z \to z_2, z_3\). Moreover, it follows from (A.8) that 
\[ q_0^2 r(z)^2 S(z)^2 D(z) = -[z^2 + 2][q - z][q - d(z)]. \]
Since \(z > z_1 > 1 > q, q - z\) is negative on the range in question. As observed above, the 
expression \(d(\cdot)\) is monotone decreasing on \((0, q_0)\), and so, in particular, on \((z_1, z_2)\).
Furthermore, a simple calculation shows that 
\[ q - d(z_j) = 2(2z_j^2 + 1)^{-1}(8z_j^2 + 16z_j)^{-1}(z_j^2 - 1)^{j/2}(2z_j^2 - 5), \]
\(j = 2, 3\) un summed. Thus, this expression will be negative at \(z_2, z_3\), positive at \(z_3\). 
We conclude that \(D(\cdot)\) has a single critical point in \((z_2, z_3)\), a minimum which 
occurs at the point \(z_0^1\) such that 
\[ d(z_0^1) = q. \]
In the case when \(q_1 < q < 1\), then the behaviour of \(T(z)\) will be determined by 
the sign of the quantity \(k - D(z_0^1) = k - k_0(q)\). If this quantity is non-positive, \(T(z)\) is 
monotone increasing on \((z_2^1, z_3^1)\). If it is positive, there will exist values \(z_1 < z < z_2\) of 
\(z\) such that 
\[ k - D(z_0^1) = 0, \quad x = 1, 2. \]

The expression \(L(\cdot)\) is negative on \((z_1, z_2)\) and positive on \((1, z_1)\) and \((z_2, \infty)\), 
so that the function \(T(z)\) will be monotone increasing on \((1, z_1)\) and \((z_2, \infty)\), monotone 
decreasing on \((z_1^1, z_2^1)\) in this case.

**APPENDIX B**

*The behaviour of \(A(\lambda, k, q)\)*

In this Appendix we study the behaviour as a function of \(\lambda\) for fixed \(k, q\,\) of the 
function \(A(\cdot)\) defined in (1.4); namely the ninth-order polynomial. 
\[ A(\lambda, k, q) = \lambda^9 - kl^7 + 2qk\lambda^4 - 3\lambda^3 - k\lambda + 2q. \tag{B.1} \]
Specifically, we shall need to identify the positive zeros of \(A\), the regions where \(A\) is 
positive and those where it is negative.

We begin by considering the special case when \(q = 1\). Notice that 
\[ A(\lambda, k, 1) = (\lambda^3 - 1)^2(\lambda^3 - k\lambda + 2). \tag{B.2} \]
Thus, for \( k < 3 \), \( A(\lambda, k, 1) \) has a single zero at \( \lambda = 1 \) and is otherwise positive. For \( k > 3 \), it has two additional zeros \( \lambda_a, \lambda_b \), with \( 0 < \lambda_a < 1 < \sqrt{(k/3)} < \lambda^* < \sqrt{k} \).

We now turn to the general case. Notice that
\[
A(1, k, q) = 2(k + 1)(q - 1),
\]
so that \( A \) is positive at \( \lambda = 1 \) for \( q > 1 \), negative for \( q < 1 \). We write \( A(\cdot) \) as a function of \( z = \lambda^2 \),
\[
\tilde{A}(z, k, q) = z^2 - 3z + 2q - k z^2 r(z) = n(z) - k z^2 r(z),
\]
where \( r(\cdot) \) was defined in Appendix A.

For \( 0 < q < 1 \), the cubic \( n(\cdot) \) has two positive zeros \( z_4, z_5 \), with \( 0 < z_4 < 1 < z_5 < \infty \). The quadratic \( r(\cdot) \) is positive definite. The function \( \tilde{A}(\cdot) \) is positive at \( \lambda = 0, \infty \), negative at \( \lambda = 1 \), and thus has at least two positive zeros. For all values of \( k \), \( \tilde{A}(\cdot) \) will be negative on \([z_4, z_5]\), so there is at least one zero in \((0, z_4)\) and at least one in \((z_5, \infty)\).

We consider the function
\[
G(z) = z^2 r(z) / n(z).
\]
The function \( G \) tends to \( \infty \) as \( z \uparrow z_4 \), \( z \uparrow z_5 \), and to zero as \( z \downarrow 0, z \uparrow \infty \). A straightforward calculation yields
\[
r(z) n(z)^2 G(z) = -[q - g_1(z)][q - g_2(z)],
\]
with
\[
g_{1.2} = \frac{1}{k} z^2 \left( 5z^4 + 10z^2 + 1 \pm (z^2 - 1)(25z^4 + 118z^2 + 1)^{1/2} \right).
\]
We find, by a further calculation, that
\[
16z^2 g_1 z = 15z^4 + 10z^2 - 1 \pm (25z^4 + 118z^2 + 1)^{1/2} (75z^6 + 211z^4 + 19z^2 + 1) \}
\]
It follows that any critical point of \( g_j, j = 1, 2 \), must satisfy the identity
\[
96z^2 [z^2 - 1]^3 [25z^2 - 1] = 0.
\]
We conclude that \( g_1 \) has a single critical point, a minimum, at \( z = 0,2 \), and that \( g_2 \) has a single critical point, a maximum at \( z = 1 \), and that these two functions are otherwise monotone. Thus, \( g_j(0) = 0, g_j \) decreases on \((0, 0.2)\) to a minimum value of \(-0.28\), and is monotone increasing on \((0.2, \infty)\), with \( g_j(1) = 1 \), \( g_j(z) \to \infty \) as \( z \to \infty \). Likewise, we see that \( g_2 \) decreases monotonically from \( \infty \) at \( z = 0 \) to a minimum value of \( -1 \) at \( z = 1 \), and is monotone increasing on \((1, \infty)\), tending to \( \infty \) as \( z \to \infty \). Since \( 0 < q < 1 \), \( q - g_1 \) will be initially positive, ultimately negative, with one zero which lies in \((0, 1)\). A simple calculation assures us that this zero is greater than \( z_4 \). Moreover, \( q - g_2 \) will always be negative. Thus, the function \( G(\cdot) \) is monotone increasing on \((0, z_4)\) and monotone decreasing on \((z_5, \infty)\), and so assumes the value \( k^{-1} \) exactly twice, once in each of these intervals. Hence \( \tilde{A}(\cdot) \) has precisely two zeros \( z_4, z_5 \) and \( 0 < z_4 < z_5 < 1 < z_5 < z^* \). The function \( \tilde{A}(\cdot) \) will be positive on \((0, z_4)\) and \((z_5, \infty)\), negative on \((z_4, z_5)\).

We introduce the notation \( z^*(q) = z^{*1} \). We now consider the case when \( q > 1 \). In this case \( n(\cdot) \) is positive on \([0, \infty)\), and \( r(\cdot) \) has two positive roots \( z_\pm \). It is clear that \( \tilde{A}(\cdot) > 0 \) on \([z_-, z_+]\). We write
\[
\tilde{A}(z, k, q) = z^2 r(z)(N(z) - k),
\]
with \( N(z) = \{G(z)\}^{-1} \). Observe that \( N(\cdot) \) is positive on \([0, z_{-})\) and \((z_{+}, \infty)\), and that \( N(z) \to \infty \) as \( z \downarrow 0 \), \( z_{+} \), and as \( z \uparrow z_{-}, \infty \). Differentiation of \( N(\cdot) \) yields (see (B.5), (B.6))

\[
\frac{1}{z^{2}r(z)^{2}}N'(z) = [q - g_{1}(z)][q - g_{2}(z)].
\]

(B.10)

Since \( q > 1 \) now, there will exist two values of \( z \) such that \( g_{2}(z) = q \). It is easy to see that one of these lies in \((0, z_{-})\) and the other in \((z_{+}, \infty)\). There exists one value such that \( g_{1}(z) = q \); this is easily shown to lie in \((1, z_{-})\).

It follows that the function \( N(\cdot) \) has local minima at \( \bar{z}_{-} \in (0, z_{-}) \) and \( \bar{z}_{+} \in (z_{+}, \infty) \). Thus, if \( k < k_{-}(q) = N(\bar{z}_{-}) \), \( \bar{A}(\cdot) \) is positive on \((0, z_{-})\); if \( k > k_{+}(q) \), \( \bar{A}(\cdot) \) will have two zeros in \((0, z_{-})\) and be negative on the subinterval between them. Similarly \( \bar{A}(\cdot) \) will have no zeros or two zeros in \((z_{+}, \infty)\) depending on whether \( k \) is less than or greater than \( k^{*}(q) = N(\bar{z}_{+}) \).

Finally, we observe that if \( q \leq 0 \), \( A(\lambda, k, q) \) has precisely one positive zero, \( \lambda_{0} \), and is positive for \( \lambda > \lambda_{0} \).